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On the number of solutions for the two-point boundary value problem on Riemannian manifolds

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Abstract

We study the solutions joining two fixed points of a time-independent dynamical system on a Riemannian manifold (M, g) from an enumerative point of view. We prove a finiteness result for solutions joining two points $p, q \in M$ that are non-conjugate in a suitable sense, under the assumption that (M, g) admits a non-trivial convex function. We discuss in some detail the notion of conjugacy induced by a general dynamical system on a Riemannian manifold. Using techniques of infinite dimensional Morse theory on Hilbert manifolds we also prove that, under generic circumstances, the number of solutions joining two fixed points is odd. We present some examples where our theory applies.

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1. Introduction

Given a second order differential equation $y'' = f(y', y, t)$ in the Euclidean plane $\mathbb{R}^2$, several authors have studied the problem of determining the number of its solutions connecting two given points (t_0, y_0) and (t_1, y_1) in the plane. This is the simplest and the oldest boundary value problem; accordingly, there is a vast literature in the context of pure and applied mathematics, as well as other sciences.

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A first natural question to ask is under which conditions there are only a finite number of solutions joining two fixed points. For instance, when the right-hand side of the equation is analytic, then the number of solutions of the boundary value problem are zeros of a suitable analytic function, hence they are a finite number if one has a priory bounds on the solutions of the problem [2].

More generally, if one fixes the initial point (t_0, y_0), then it is interesting to study how the number of solutions varies when the final endpoint (t_1, y_1) varies in the plane, and what kind of decomposition of the plane $\mathbb{R}^2 = \bigcup_{i=0}^{\infty} \sigma_i$ is obtained, where σ_i is defined as the set of those points (t_1, y_1) for which there are exactly i solutions of the problem.

This decomposition, called *focal decomposition* (or also *σ-decompositions*) by some authors [20] was introduced in [18] and developed by several other authors (see [14,20] and the references therein). The deepest result proven concerning focal decompositions [20] is that, under suitable technical assumptions, each σ_i is the union of strata of an analytic Whitney stratification, and that σ_i has empty interior when i is even. Many techniques involved do not generalize to the case of higher dimensional systems of differential equations (see for instance [20, Section 8]), and there is not much literature on the topic concerning higher dimensional systems.

Another direction of investigation consists in studying similar boundary value problems in curved spaces; it is absolutely evident that both the topology and the metric (i.e., the curvature) of the configuration space has a deep influence on the number of solutions of a two-point boundary value problem. In this paper we study the solutions of autonomous dynamical system on a Riemannian manifold of the form $(D/\mathrm{d}t)\dot{x} = -\nabla V(x)$; interpreting M as the configuration space of a mechanical system, then these solutions represent the trajectories of masses under the influence of a conservative force with potential $-V$. We fix an initial value $p = x(0)$ in M, a positive value of the time parameter T, and we look at the decomposition of M into the sets σ_i, $i = 0, \dots, +\infty$, where σ_i consists of all points of M that can be reached after a time T by exactly i distinct trajectories of the dynamical system starting at the instant $t = 0$ in p.

Our first aim is to establish sufficient conditions on (M, g) and V to guarantee that σ_∞ has null measure in M, i.e., that for almost all choice of q, the number of trajectories of the dynamical system that start at p and terminate at q after a fixed time is finite.

In the case of non-flat metrics, the problem is interesting also in the case that $V = 0$, i.e., when the dynamical system reduces to the geodesic equation in (M, g). For instance, finiteness results for Riemannian geodesics between two fixed points give analogous results for lightlike geodesics between a point and an observer of a (conformally) stationary Lorentzian manifold. This kind of results can be applied in Astrophysics [11,12] to obtain information on the so-called *gravitational lensing effect* in General Relativity, which produces the phenomenon that an astronomer observes multiple images of the same light (or radio) source [22].

Our main finiteness results (Section 3) use an assumption that can be considered both topological and metrical on (M, g), namely, we prove that if (M, g) admits a non-trivial proper convex function which is non-increasing on the flow lines of the gradient of V, then the number of trajectories $x : [0, T] \to M$ of the dynamical system joining two points p and q is finite for almost all pairs p and q. Moreover, we prove that under suitable boundedness on the growth of the potential V at infinity, such number is generically odd, i.e., the sets σ_{2i} defined above have null measure for all $i = 1, \ldots, +\infty$.

In order to clarify the result, it may be helpful to discuss a simple but instructive example that was our initial motivation for the convexity assumption. Let us look at the case $V = 0$ and the corresponding dynamical systems, whose solutions are geodesics in (M, g). Probably the first finiteness result for this situation is the well known theorem of Hadamard, that states that if M is simply connected, (M, g) is complete and it has negative sectional curvature everywhere, then between any two points there exists a *unique* geodesic. In this situation, for each point $p \in M$, the map $F(x) = \mathrm{dist}(p, x)$ is *strictly convex* on M, i.e., $F \circ \gamma : \mathbb{R} \to \mathbb{R}$ is strictly convex for all non-constant geodesic $\gamma : \mathbb{R} \to M$. On the other hand (strictly), convex functions exist on a larger class of (non-compact) Riemannian manifolds (see Appendix A), and it is possible to prove that, in manifolds that admit strictly convex proper functions there are only a finite number of geodesics between two non-conjugate points [11]. In this paper we exploit the methods of [11,12] to treat the case of general conservative dynamical systems; the main idea is that solutions of a general conservative dynamical system remaining inside a compact subset and with diverging energy *tend* to geodesics in a suitable sense (see Lemma 3.1 and the proof of Proposition 3.2), i.e., the gravitational forces have a neglectible effect on masses with very large kinetic energy, whose trajectories tend to be *straight* as the kinetic energy goes to infinity. In this situation, in order to obtain finiteness results one can apply the standard convexity techniques.

Observe that the existence of convex functions depend crucially on the metric (that defines the geodesics), but also on the topology of M: it is well known that compact manifolds do not admit non-constant convex functions, and that a manifold that admits a strictly convex function with a minimum point is diffeomorphic to $\mathbb{R}^n$. A short survey of the basic properties, examples and constructions with convex functions in Riemannian manifolds is presented in Appendix A.

It turns out that, as in the geodesic case, the points that are *conjugate* relatively to a dynamical system play a crucial role in the theory. Roughly speaking, two points p and q on M are conjugate relatively to a dynamical system (see Definition 2.1) if there exist a homotopy $\{x_s\}_{s\in]-\varepsilon,\varepsilon[}$ of solutions of the dynamical system satisfying the boundary conditions $x_s(0) = p$ and $x_s(T) = q$, up to infinitesimal of order greater than one. However, it must be noted that, unlike the case of geodesics, the set of solutions of a general dynamical system is not invariant by arbitrary affine reparameterizations, so that the notions of conjugacy, *Jacobi fields* (Definition 2.2) and of *exponential map* (see the proof of Proposition 2.6) relative to a dynamical system require special attention. A detailed presentation of the main properties of conjugate points relative to a dynamical system is given in Section 2; among other things, we prove that the set of points that are conjugate to a given one has null measure (Proposition 2.6).

The finiteness problem for solutions of the dynamical system satisfying the two-point boundary condition is studied in Section 3; we first give a *local* finiteness result (Proposition 3.2), and then, using further assumptions on the convex function, we prove the *global* finiteness result for solutions joining two non-conjugate points (Proposition 3.4). As a matter of fact, passing from the local to the global result is possible for those dynamical system having the properties that all their solutions with endpoints inside a compact set do not leave a possibly larger compact subset of M. Such differential equations are defined to be *regular* in some references (see for instance [20]); observe that the regularity plays a crucial role also in many results of [20]. In this language, the existence of a strictly convex function on M that is non-increasing on the flow lines of ∇V is a sufficient condition that guarantees the regularity of the dynamical system. Other regularity criteria, as well as examples where our theory applies are discussed in Section 4.

Finally, in Section 5 we apply techniques of Critical Point Theory, more precisely the *Morse theory* and the *Ljusternik–Schnirelman theory*, to study the *parity* of the number of solutions between two non-conjugate points (Proposition 5.1), and to show that if M is not contractible then the number of solutions between any two points is never finite (Proposition 5.2). These theories require some technical assumptions on the variational setup; for this reason we assume a suitable control on the growth of the potential V at infinity.

2. Dynamical systems on Riemannian manifolds: conjugate points and focal decomposition

In this section we introduce and discuss briefly the main notions about conservative dynamical systems whose configuration space is a Riemannian manifold. Some of the results presented here are not new, but they are nevertheless stated and proved in our context for the reader's convenience.

Let (M, g) be an n-dimensional Riemannian manifold; we will denote by ∇ the covariant derivative of the Levi–Civita connection of g and by R the curvature tensor of ∇, chosen with the following sign convention:

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

For a vector field v along a curve x in M, we will use both notations v' and $(D/\mathrm{d}t)v$ to denote the covariant derivative of v along x; the latter symbol being used almost exclusively to denote the covariant derivative $(D/\mathrm{d}t)\dot{x}$ of the tangent field $\dot{x}$. The symbol $\|\cdot\|$ will denote the *norm* of tangent vectors to M induced by g. For the basics of the geometry of Riemannian manifolds we refer to the textbook [5].

Given a sufficiently regular map $f : M \to \mathbb{R}$, let us denote by ∇f and Hess f, respectively the gradient and the Hessian of f. Recall that ∇f is the vector field in M defined by the relation $g(\nabla f(p), \cdot) = \mathrm{d} f(p)$, and that Hess f is the $(2, 0)$-tensor field whose value at the point $p \in M$ is a symmetric bilinear form on T_pM defined by

$$\mathrm{Hess}_p f(v, w) = g(\nabla_v \nabla f(p), w), \quad v, w \in T_pM.$$

Using the metric g, we will also think of the Hessian as a g-symmetric linear endomorphism $\mathrm{Hess}_p f : T_pM \to T_pM$ defined by

$$g(\mathrm{Hess}_p f(v), w) = \mathrm{Hess}_p f(v, w), \quad \forall\, v, w \in T_pM.$$

2.1. Conservative dynamical systems on M

Let $V : M \to \mathbb{R}$ be a map of class C^2 and consider the following differential equation for curves in M:

$$\frac{D}{\mathrm{d}t}\dot{x} = -\nabla V(x). \tag{2.1}$$

In this section we will discuss the notion of conjugate points along a solution of (2.1); this notion generalizes the classical notion of conjugate points along a geodesic in Riemannian or semi-Riemannian geometry. We will refer to Eq. (2.1) as a *dynamical system* on M with *potential* $-V$.

Since (2.1) is autonomous, then the set of its solutions is invariant by time-translation and by time-reversing reparameterizations. Note however that, unless V is locally constant, the set of solutions of (2.1) is *not* invariant by arbitrary affine reparameterizations.

For this reason, in our setup we will choose an interval of the form $[0, T]$ as the basic domain of our curves, and, unlike the case of conjugate points in semi-Riemannian geometry, the notion of conjugate points induced by (2.1) will be dependent on the choice of such interval.

An immediate computation shows that the solutions of (2.1) satisfy the following conservation law:

$$\tfrac{1}{2}g(\dot{x}, \dot{x}) + V(x) \equiv E_x \quad \text{(constant)}. \tag{2.2}$$

It is well known that the solutions of (2.1) that satisfy the boundary conditions:

$$x(0) = p, \quad x(T) = q, \tag{2.3}$$

with $p, q \in M$ are precisely the stationary points of the Lagrangian action functional:

$$\mathcal{L}_T(x) = \int_0^T \left[\frac{1}{2}g(\dot{x}, \dot{x}) - V(x)\right] \mathrm{d}t, \tag{2.4}$$

defined in the smooth Hilbert manifold $\Omega_{p,q}([0, T], M)$ of all curves $x : [0, T] \to M$ of class H^1 in M such that $x(0) = p$ and $x(T) = q$. Since (M, g) is complete, then also $\Omega_{p,q}([0, T], M)$ is a complete Hilbert manifold. The functional $\mathcal{L}_T$ is of class C^2 on $\Omega_{p,q}([0, T], M)$. Given a critical point $x \in \Omega_{p,q}([0, T], M)$ of $\mathcal{L}_T$, the second variation $\mathrm{d}^2\mathcal{L}_T(x)$ of $\mathcal{L}_T$ at x is easily computed to be the following bounded symmetric bilinear form:

$$\mathrm{d}^2\mathcal{L}_T(x)(v, w) = I(v, w) = \int_0^T [g(v', w') + g(R(\dot{x}, v)\,\dot{x}, w) - \mathrm{Hess}_x\, V(v, w)]\,\mathrm{d}t, \tag{2.5}$$

defined in the Hilbert space $T_x\Omega_{p,q}([0, T], M)$, which consists of all H^1 vector fields along x vanishing at 0 and at T.

Our main goal is to study the number of solutions of (2.1) that satisfy the boundary conditions (2.3); for all $i = 0, 1, \ldots, +\infty$, we set:

$$\sigma_i = \{q \in M : \text{there exist precisely } i \text{ solutions of (2.1) and (2.3)}\}. \tag{2.6}$$

Obviously, the σ_i's, $i = 0, \ldots, +\infty$ are pairwise disjoint and they form a partition of M.

2.2. Conjugate points defined by (2.1)

Recall that a critical point x for the action functional $\mathcal{L}_T$ in $\Omega_{p,q}([0,T],M)$ is said to be *non-degenerate* if the second variation $\mathrm{d}^2\mathcal{L}_T(x)$ is a strongly non-degenerate bilinear form on $T_x\Omega_{p,q}([0,T],M)$, i.e., when the self-adjoint operator on $T_x\Omega_{p,q}([0,T],M)$ that represents $\mathrm{d}^2\mathcal{L}_T(x)$ is an isomorphism. The critical point x is said to be *degenerate* if it is not non-degenerate.

Definition 2.1. A point $q \in M$ is said to be *conjugate to p on* $[0,T]$ *relatively to* (2.1) if there exists a degenerate critical point x of $\mathcal{L}_T$ in $\Omega_{p,q}([0,T],M)$.

By the time-orientation invariance, the notion of conjugacy on $[0,T]$ is clearly symmetric in p and q, so that, when q is conjugate to p on $[0,T]$ relatively to (2.1), we will say that p and q are conjugate points.

The notion of conjugacy can be given in terms of solutions of the linearization of (2.1) is as follows.

Definition 2.2. A *V-Jacobi field* along the solution x is a vector field J along x that satisfies the linearization of equation (2.1), which is easily computed as:

$$J'' - R(\dot{x}, J)\dot{x} + \mathrm{Hess}_x\, V(J) = 0. \tag{2.7}$$

Thus, by definition, V-Jacobi fields along a solution x are variational vector fields that correspond to variations of x by solutions $x_s : [0,T] \to M$ of (2.1); here, by *variations* of x by solutions of (2.1) we mean a family $s \mapsto x_s$ of solutions of (2.1) such that $x_0 = x$ and such that $s \mapsto x_s(0)$ is of class C^2. This statement is made more precise in the following lemma.

Lemma 2.3. *A C^2 vector field J along a solution $x : [0,T] \to M$ of* (2.1) *is a V-Jacobi field iff there exists a variation* $\{x_s\}_{s\in]-\varepsilon,\varepsilon[}$ *of x by solutions of* (2.1) *in* $[0,T]$ *with variational vector field J, i.e.* $(\mathrm{d}/\mathrm{d}s)x_s(t)|_{s=0} = J(t)$ *for all* $t \in [0,T]$.

Proof. Clearly, variational vector fields corresponding to variations by solutions of x satisfy the linearized equation (2.7). Conversely, assume that J satisfies (2.7), let $\varepsilon > 0$ be small enough, let $\gamma :]-\varepsilon,\varepsilon[\to M$ be a smooth curve satisfying $\gamma(0) = x(0)$ and $\dot{\gamma}(0) = J(0)$ and let $W :]-\varepsilon,\varepsilon[\to TM$ be a smooth vector field along γ such that $W(0) = \dot{x}(0)$ and such that $W'(0) = J'(0)$. For each s, let $t \mapsto x_s(t)$ denote the maximal solution of (2.1) in M satisfying $x_s(0) = \gamma(s)$ and $(\mathrm{d}/\mathrm{d}t)x_s|_{t=0} = W(s)$. Clearly, $x_0 \equiv x$ on $[0,T]$, and by continuity, for $\varepsilon > 0$ small enough, x_s is defined on $[0,T]$. Let $\tilde{J}$ denote the V-Jacobi variational field along x corresponding to $\{x_s\}$; one has $\tilde{J}(0) = \dot{\gamma}(0) = J(0)$ and $\tilde{J}'(0) = W'(0) = J'(0)$, so that $\tilde{J} = J$ and we are done. □

By an easy boot-strap argument and integration by parts in (2.5) one proves the following lemma.

Lemma 2.4. *Let $x \in \Omega_{p,q}([0,T],M)$ be a critical point of $\mathcal{L}_T$. Then, a vector field $J \in T_x\Omega_{p,q}([0,T],M)$ is a V-Jacobi field along x if and only if J is in the kernel $\mathrm{Ker}(I)$ of I, i.e., if and only $I(J,w)=0$ for all $w \in T_x\Omega_{p,q}([0,T],M)$.*

We are now ready to prove the relation between conjugate points and V-Jacobi fields.

Proposition 2.5. *A point $q \in M$ is conjugate to p on $[0,T]$ relatively to (2.1) if and only if there exists a solution $x:[0,T]\to M$ of (2.1) satisfying (2.3) and a non-zero V-Jacobi field J along x such that $J(0)=J(T)=0$.*

Proof. If there exists a non-zero V-Jacobi field J along x with $J(0)=J(T)=0$, then $J \in T_x\Omega_{p,q}([0,T],M)$; moreover by Lemma 2.4 such Jacobi field would be a non-zero element in $\mathrm{Ker}(I)$. This implies that x is a degenerate critical point of $\mathcal{L}_T$ in $\Omega_{p,q}([0,T],M)$, and so p and q are conjugate.

On the other hand, assume that p and q are conjugate on $[0,T]$ relatively to (2.1); then, by definition, there exists a degenerate critical point x of $\mathcal{L}_T$ in $\Omega_{p,q}([0,T],M)$, i.e., a point x for which the bilinear form I of (2.5) is not strongly non-degenerate. Now, we claim that weak and strong non-degeneracy are equivalent for the bilinear form (2.5), i.e., that I is strongly non-degenerate if and only if $\mathrm{Ker}(I)=\{0\}$. To see this, observe that I is represented on the Hilbert space $T_x\Omega_{p,q}([0,T],M)$ by a Fredholm operator of index zero. Namely, the bilinear form $(v,w)\mapsto \int_0^T g(v',w')\,\mathrm{d}t$ is represented by a positive isomorphism of $T_x\Omega_{p,q}([0,T],M)$, which is a Fredholm operator of index 0. Moreover, the bilinear form

$$(v,w)\mapsto \int_0^T [g(R(\dot{x},v)\dot{x},w) - \mathrm{Hess}\, V(v,w)]\,\mathrm{d}t$$

is continuous in the C^0 topology, and by the compact inclusion of C^0 in H^1 it follows that it is represented by a compact self-adjoint operator on $T_x\Omega_{p,q}([0,T],M)$. Since compact perturbations of Fredholm operators are still Fredholm operators with the same index, it follows that I is represented by a Fredholm operator of index zero.

So, I is strongly non-degenerate if and only if $\mathrm{Ker}(I)=\{0\}$, and the conclusion follows easily from Lemma 2.4. □

We have now our setup ready to prove the genericity of the non-conjugacy condition. The technique is analogue to the case of conjugacy by geodesics in Riemannian manifolds: the conjugate points are characterized as *critical values* of a differentiable map (the analogue of the Riemannian exponential map) and the conclusion is obtained as an application of Sard's theorem.

Proposition 2.6. *Let $p \in M$ be fixed. The set of points $q \in M$ that are conjugate to p on $[0,T]$ relatively to (2.1) has null measure in M.*

Proof. For $p \in M$, we introduce a C^1 map $\mathcal{F}_p : A \subset T_pM \to M$ defined on an open neighborhood of 0 in T_pM by setting

$$\mathcal{F}_p(v) = x(T),$$

where $x : [0, T] \to M$ is the unique solution of (2.1) with $x(0) = p$ and $\dot{x}(0) = v$. The differentiability of $\mathcal{F}_p$ is standard: as in the case of the exponential map in Riemannian geometry, $\mathcal{F}_p$ is obtained from the flow of some C^1 vector field in TM, which is of class C^1.

The vector $v \in A$ is a critical point for $\mathcal{F}_p$, i.e., the differential $\mathrm{d}(\mathcal{F}_p)_v : T_pM \to T_{\mathcal{F}_p(v)}M$ is not surjective, iff $x(T) = \mathcal{F}_p(v)$ is conjugate to p along x. Namely, using Lemma 2.3, it is easily computed:

$$\mathrm{d}(\mathcal{F}_p)_v(w) = J(T),$$

where J is the unique V-Jacobi field along x such that $J(0) = 0$ and $J'(0) = w$. Thus, $\mathrm{d}(\mathcal{F}_p)_v$ is not surjective iff $\mathrm{Ker}(\mathrm{d}(\mathcal{F}_p)_v) \neq \{0\}$, and by Proposition 2.5 this is equivalent to $\mathcal{F}_p(v)$ being conjugate to p.

So, the points q that are conjugate to p are the critical values of $\mathcal{F}_p$, and the conclusion follows from Sard's theorem. □

2.3. *Conjugate points along a solution of (2.1)*

We fix a solution $x : [0, T] \to M$ of (2.1) and we introduce the notion of a conjugate point (or better, of conjugate instant) along x is as follows.

Definition 2.7. Let $x : [0, T] \to M$ be a solution of (2.1). An instant $t_0 \in]0, T]$ in M is said to be *conjugate* along x if there exists a non-zero V-Jacobi field J along x with $J(0) = J(t_0) = 0$. The *multiplicity* $\mathrm{mul}(t_0)$ of the conjugate instant t_0 is defined to be the dimension of the space of V-Jacobi fields along x such that $V(0) = V(t_0) = 0$; clearly, $\mathrm{mul}(t_0) \leq n$.

By Proposition 2.5, an instant $t_0 \in]0, T]$ is conjugate along x if and only if the point $x(t_0)$ is conjugate to $x(0)$ on $[0, t_0]$ relatively to (2.1), being $x|_{[0,t_0]}$ the corresponding degenerate critical point of $\mathcal{L}_{t_0}$ in $\Omega_{x(0),x(t_0)}([0, t_0], M)$.

Given a solution $x : [0, T] \to M$ of (2.1), we will denote by $\mathbb{J}_x$ the n-dimensional vector space of all V-Jacobi fields J along x satisfying $J(0) = 0$. For all $t_0 \in]0, T]$, let

$$\phi_{t_0} : \mathbb{J}_x \to T_{x(t_0)}M, \tag{2.8}$$

denote the evaluation map $J \mapsto J(t_0)$; observe that the fact that t_0 is a conjugate instant along x if and only if ϕ_{t_0} is *not* an isomorphism. Moreover, if t_0 is a conjugate instant along x, then its multiplicity equals the codimension of the image $\mathrm{Im}(\phi_{t_0})$ in $T_{x(t_0)}M$.

The following result is totally analogous to the case of Jacobi fields along a semi-Riemannian geodesic.

Lemma 2.8. *If J_1, J_2 are V-Jacobi fields along x then the quantity $g(J_1', J_2) - g(J_1, J_2')$ is constant. In particular, if $J_1, J_2 \in \mathbb{J}_x$, then $g(J_1', J_2) - g(J_1, J_2') \equiv 0$.*

Proof. Using (2.7) and the symmetry of the curvature tensor and of the Hessian operator one computes immediately:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(g(J_1', J_2) - g(J_1, J_2'))\\&\quad = g(J_1'', J_2) - g(J_1, J_2'')\\&\quad = g(R(\dot{x}, J_2)\dot{x}, J_1)) - \mathrm{Hess}\, V(J_1, J_2) - g(R(\dot{x}, J_1)\dot{x}, J_2)) + \mathrm{Hess}\, V(J_2, J_1) = 0.\end{aligned}$$

□

Using the conservation law of Lemma 2.8 satisfied by the V-Jacobi fields it is possible to adapt the classical proof of discreteness of the set of conjugate points along a Riemannian geodesic to prove the following proposition.

Proposition 2.9. *Let $x : [0, T] \to M$ be a solution of* (2.1); *then, the set of conjugate instants along x is finite.*

Proof. Let $t_0 \in]0, T]$ be a conjugate instant along x; to prove the proposition we will show that there are no other conjugate instants in a neighborhood of t_0. To this aim, set $k = \mathrm{mul}(t_0)$ and choose $J_1, \ldots, J_n$ a basis of the space $\mathbb{J}_x$.

Clearly, the vectors $J_{k+1}(t_0), \ldots, J_n(t_0)$ form a basis of $\mathrm{Im}(\phi_{t_0})$; moreover, their derivatives $J_1'(t_0), \ldots, J_k'(t_0)$ form a basis for the orthogonal $\mathrm{Im}(\phi_{t_0})^\perp$. To see this, observe that these k vectors in $T_{x(t_0)}M$ are linearly independent, because the vectors $(J_i(t_0), J_i'(t_0))$, $i = 1, \ldots, k$ are linearly independent and $J_i(t_0) = 0$ for $i = 1, \ldots, k$. To prove the claim now simply observe that, by Lemma 2.8, $J_1'(t_0), \ldots, J_k'(t_0)$ do indeed belong to the orthogonal space $\mathrm{Im}(\phi_{t_0})^\perp$, because:

$$\begin{aligned}g(J_i'(t_0), J_j(t_0)) &= g(J_i'(t_0), J_j(t_0)) + g(J_i(t_0), J_j'(t_0))\\&= g(J_i'(0), J_j(0)) + g(J_i(0), J_j'(0)) = 0,\end{aligned}$$

for all $i = 1, \ldots, k$ and all $j = k + 1, \ldots, n$. Since $T_{x(t_0)}M = \mathrm{Im}(\phi_{t_0}) \oplus \mathrm{Im}(\phi_{t_0})^\perp$, we conclude that the family

$$J_1'(t_0), \ldots, J_k'(t_0), J_{k+1}(t_0), \ldots, J_n(t_0)$$

is a basis of $T_{x(t_0)}M$.

We now define a family $\{\tilde{J}_i\}_{i=1}^n$ of C^2 vector fields along x by setting $\tilde{J}_i = J_i$ for $i = k + 1, \ldots, n$, and

$$\tilde{J}_i(t) = \begin{cases} \dfrac{J_i(t)}{t - t_0} & \text{for } t \neq t_0, \\ J'(t_0) & \text{if } t = t_0, \end{cases} \qquad i = 1, \ldots, k.$$

An instant $t \neq t_0$ in $]0, T]$ is conjugate along x iff $h(t) = \det(\tilde{J}_1(t), \ldots, \tilde{J}_n(t)) = 0$ because for $t \neq t_0$ the vanishing of this determinant is equivalent to the vanishing of $\det(J_1(t), \ldots, J_n(t)) = 0$. Now, since the $\tilde{J}_i(t_0)$'s are linearly independent, it is $h(t_0) \neq 0$, and by continuity $h(t) \neq 0$ in a neighborhood of t_0. This concludes the proof. □

The notion of conjugate instants along a solution of (2.1) is related to the Morse index of the action functional $\mathcal{L}_T$ at the critical point x. Recall that given a C^2 map F on a differentiable manifold X and a critical point $x \in X$ of F, then the *Morse index* of F at x is the index of the second variation $\mathrm{d}^2 F(x)$ on $T_x X$, which is the supremum of the dimensions of the subspaces of $T_x X$ on which the symmetric bilinear form $\mathrm{d}^2 F(x)$ is negative definite. We state now an extension to a solution of (2.1) of the classical Morse Index Theorem for geodesics joining two points on a Riemannian manifold.

Proposition 2.10. *Let $x : [0, T] \to M$ be a solution of (2.1) satisfying (2.3), i.e., x is a critical point of the functional $\mathcal{L}_T$ (2.4) in $\Omega_{p,q}([0, T], M)$. Then the Morse index of $\mathcal{L}_T$ at x is finite, and it is equal to the number of conjugate instants along x in $]0, T[$ counted with multiplicity.*

The proof is quite similar to the proof of the Morse Index Theorem for Riemannian geodesics (see for instance [8, Theorem 2.4], or the books [13,16]). An extension of the index theorem has been recently obtained in [21] in the case of *non-convex* Hamiltonian systems.

3. Finiteness of the number of solutions for the two-point boundary value problem

In this section we will prove the finiteness of the number of solutions of (2.1) joining two non-conjugate points p and q in a Riemannian manifold (M, g) that admits non-trivial convex functions.

We start with a technical lemma concerning the C^2 convergence of sequences of curves whose covariant acceleration goes to zero uniformly. The result is trivial when the metric is flat, i.e., when the covariant acceleration coincides with the second derivative. For the general case the argument is more delicate, due to the fact that for the use of local coordinates one first needs to prove the uniform convergence of the sequence.

Lemma 3.1. *Let (M, g) be a complete Riemannian manifold, let $y_n : [a, b] \to M$ be a sequence of C^2 maps such that $\dot{y}_n(a)$ converges to v_0 in TM and such that $\|(D/\mathrm{d}t)\dot{y}_n\|$ converges to 0 uniformly in $[a, b]$. Then, y_n converges in the C^2 topology to the (affinely parameterized) geodesic $y : [a, b] \to M$ with $y'(t_0) = v_0$.*

Proof. We will show first that $\|\dot{y}_n\|$ is uniformly bounded in $[a, b]$; to this aim, set $u_n = \|(D/\mathrm{d}t)\dot{y}_n\|$, denote by $\delta_n = g(\dot{y}_n, \dot{y}_n)$ and observe that $|\delta'_n| \le 2u_n\sqrt{\delta_n} \le 2u_n(\delta_n + 1)$. It follows from Gronwall's inequality that:

$$\delta_n \le K\,\mathrm{e}^{k(b-a)},$$

where $K = \sup_n[\delta_n(a) + 2\int_a^b u_n\,\mathrm{d}t]$ and $k = 2\sup_n \|u_n\|_\infty$. Hence, $\|\dot{y}_n\|$ is uniformly bounded; set $C = \sup_n \|\dot{y}_n\|_\infty$.

Let $y : [a, b] \to M$ be the geodesic with $\dot{y}(a) = v_0$ and let $r > 0$ be a *normal radius* for all the points of the image of y, i.e., the exponential map $\exp_{y(t)}$ is a diffeomorphism on the ball of radius r centered at the origin of $T_{y(t)}M$ for all $t \in [a, b]$. Choose $\varepsilon > 0$ with $C\varepsilon < r$ and with $\|v_0\|\varepsilon < r$.

We make the following claim: if $\dot{y}_n(t_0)$ converges to $\dot{y}(t_0)$ for some $t_0 \in [a, b]$, then y_n converges in the C^2 topology to y on the interval $[t_0 - \varepsilon, t_0 + \varepsilon] \cap [a, b]$. The proof of the Lemma will follow immediately from the claim, since $\dot{y}_n(a)$ converges to $v_0 = \dot{y}(a)$.

Let U be the geodesic ball centered at $y(t_0)$ and of radius r; observe that U is the domain of a normal geodesic coordinate system. Since $\|v_0\|\varepsilon < r$, it follows that $y|_{[t_0-\varepsilon,t_0+\varepsilon]\cap[a,b]}$ has image in U; since $C\varepsilon < r$ and $y_n(t_0)$ tends to $y(t_0)$, it follows that $y_n|_{[t_0-\varepsilon,t_0+\varepsilon]\cap[a,b]}$ has image in U for all n sufficiently large. Set $z_n = (D/\mathrm{d}t)\dot{y}_n$ and using local coordinates in U, for n large enough we can write

$$(\ddot{y}_n)^k + \sum_{i,j} \Gamma^k_{i,j}(y_n)(\dot{y}_n)^i(\dot{y}_n)^j = (z_n)^k,$$

where $\Gamma^k_{i,j}$ denote the Christoffel symbols of the Levi–Civita connection of g in the local chart. Since z_n tends uniformly to 0 in $[t_0 - \varepsilon, t_0 + \varepsilon] \cap [a, b]$, $y_n(t_0)$ tends to $y(t_0)$ and $\dot{y}_n(t_0)$ tends to $y(t_0)$, standard results on the continuous dependence of the solution of an initial value problem from the data allow to conclude that y_n tends to y in the C^2 topology on the given interval. This proves the claim and concludes the proof. □

We can now prove that, if (M, g) admits a strictly convex function (see Definition A.1), then there are only a finite number of solutions of (2.1) and (2.3) that remain inside a compact subset of M.

In order to prove the result, we need to recall a few basic facts from Critical Point Theory. We recall that, given a C^1 functional $f : X \to \mathbb{R}$ on a Hilbert manifold $(X, \mathfrak{h})$, then f satisfies the *Palais–Smale condition at level* $c \in \mathbb{R}$ if every sequence (x_k) in X such that

$$\lim_{k\to\infty} f(x_k) = c, \qquad \lim_{k\to\infty} \mathfrak{h}(\nabla f(x_k), \nabla f(x_k)) = 0, \tag{3.1}$$

has a subsequence converging in X. By $\nabla f(x)$ we denote the *gradient* of f at $x \in X$, which is a bounded linear operator on the Hilbert space T_xX defined by $\mathfrak{h}(\nabla f(x), \cdot) = \mathrm{d}f(x)$. A sequence (x_k) satisfying (3.1) is called a *Palais–Smale sequence* for f in X at the level c. If f is of class C^2, then f is said to be a *Morse function* if all its critical points are non-degenerate. For instance, we have proven in Proposition 2.5 that the action functional (2.4) is a Morse function on $\Omega_{p,q}([0, T], M)$ provided that p and q are not conjugate on $[0, T]$ relatively to (2.1).

It is well known that, as in the case of smooth functions on a finite dimensional differentiable manifold, if f is a Morse function on the Hilbert manifold $(X, \mathfrak{h})$ then each critical point of f is isolated.

Proposition 3.2. *Let (M, g) be a complete Riemannian manifold, $V : M \to R$ a map of class C^2 and assume that M admits a strictly convex function $F : M \to \mathbb{R}$. Consider the dynamical system* (2.1) *with initial conditions* (2.3); *assume that p and q are not conjugate on* $[0, T]$ *relatively to* (2.1). *Then, for all compact subset $K \subset M$ with $p, q \in K$, there are only a finite number of solutions with image in K.*

Proof. Let $K \subset M$ be a fixed compact set and assume that $(x_k)_{k=1}^{\infty}$ is a sequence of pairwise distinct solutions of (2.1) and (2.3) having images in K. The proof of the proposition will be obtained by showing that p and q must be conjugate relatively to (2.1) on $[0, T]$. To this aim, since the image of the x_k's remain in the compact set K, we can assume that V is bounded on M, and therefore that $\mathcal{L}_T$ satisfies the Palais–Smale condition at every level on $\Omega_{p,q}([0, T], M)$ as we can see, for instance, by the methods developed in [15].

Indeed, it is well known that $\mathcal{L}_T$ satisfies the Palais–Smale condition when V has *sub-quadratic growth*, i.e., when there exists constants $C_1, C_2 \in \mathbb{R}^+$ and $\alpha \in]0, 2[$, such that $V(x) \le C_1 + C_2 \cdot \mathrm{dist}(x, x_0)^{\alpha}$ for all $x \in M$ and for some fixed x_0 in M.

Now, assume by contradiction that p and q are not conjugate; then it must be:

$$\lim_{k\to\infty} \mathcal{L}_T(x_k) = +\infty, \tag{3.2}$$

for, otherwise (x_k) would contain a Palais–Smale subsequence at the level

$$c = \limsup_{k\to\infty} \mathcal{L}_T(x_k) < +\infty$$

(recall that each x_k is a critical point of $\mathcal{L}_T$), and therefore it would have a converging subsequence. Since $\mathcal{L}_T$ is a Morse function on $\Omega_{p,q}([0, T], M)$ when p and q are not conjugate, the lack of discreteness of the critical points of $\mathcal{L}_T$ would then imply that p and q are conjugate and (3.2) is proven.

Now, recall that each x_k satisfies the conservation law (2.2), and for each $k \in \mathbb{N}$ set:

$$E_k = \tfrac{1}{2} g(\dot{x}_k, \dot{x}_k) + V(x_k).$$

An immediate calculation gives:

$$E_k = \frac{1}{T}\mathcal{L}_T(x_k) + 2\int_0^T V(x_k(t))\,\mathrm{d}t,$$

and since V is bounded in K from (3.2) we obtain that

$$\lim_{k\to\infty} E_k = +\infty.$$

Again, by the boundedness of V from the above equality we get that $g(\dot{x}_k, \dot{x}_k)$ is *uniformly divergent* on $[0, T]$, i.e.,

$$\lim_{k\to\infty} [\min_{t\in[0,T]} g(\dot{x}_k(t), \dot{x}_k(t))] = +\infty.$$

For each $k \in \mathbb{N}$, set:

$$b_k = \|\dot{x}_k(\tfrac{1}{2}T)\|,$$

so that

$$\lim_{k\to\infty} b_k = +\infty, \tag{3.3}$$

moreover we denote by $y_k : [-b_k, b_k] \to M$ the curve:

$$y_k(s) = x_k\left(\frac{T}{2} + s \cdot \frac{T}{2b_k}\right), \qquad s \in [-b_k, b_k].$$

One computes immediately

$$\dot{y}_k(0) = \dot{x}_k\left(\frac{T}{2}\right)\cdot\frac{T}{2b_k}, \qquad \|\dot{y}_k(0)\| \equiv \frac{T}{2}, \tag{3.4}$$

and

$$\frac{D}{\mathrm{d}s}\dot{y}_k(s) = \frac{D}{\mathrm{d}s}\dot{x}_k\left(\frac{T}{2} + s\cdot\frac{T}{2b_k}\right)\cdot\frac{T^2}{4b_k^2}. \tag{3.5}$$

From (3.4) we obtain that, up to subsequences, we can assume that $\dot{y}_k(0)$ converges to a non-zero vector $v_0 \in TM$, and since

$$\left\|\frac{D}{\mathrm{d}t}\dot{x}_k\right\| = \|\nabla V(x_k)\|$$

is bounded, from (3.3) and (3.5) we get that the norm of the covariant derivative $(D/\mathrm{d}t)\dot{y}_k$ is uniformly convergent to 0 on each compact subset of $\mathbb{R}$.

We now use Lemma 3.1 to conclude that y_k converges in the C^2 topology on each bounded interval of $\mathbb{R}$ to a non-constant affinely parameterized geodesic $y : \mathbb{R} \to M$. Since y_k has image in K for all $k \in \mathbb{N}$, it follows that y has image in K and in particular $F(y(s))$ is bounded. But the function $F \circ y : \mathbb{R} \to \mathbb{R}$ is strictly convex by assumption, and it cannot be bounded.

We therefore get to a contradiction and the proposition is proven. □

Remark 3.3. It is not clear whether the non-conjugacy assumption for the points p and q can be omitted in Proposition 3.2, even though the proof presented only works in this case. The authors do not know any counterexample to the finiteness result of Proposition 3.2 for the number of solutions joining two conjugate points, and it could be conjectured that, possibly under stronger assumptions, one has a finite number of solutions also in the case of boundary value problems with conjugate endpoints. This seems to be the case in a number of circumstances in the case of one-dimensional problems [2,20].

We will now focus our attention to the problem of determining under which conditions we have finiteness of the number of *all* solutions of (2.1) and (2.3). The central result in this direction is based on a suitable assumption relating the growth at infinity of the convex function F along the flow lines of the potential V. In order to present the result in a clearer form, we will first state a version of the theorem that uses non-optimal assumptions, and later (Remark 3.5) we will describe how to weaken the hypotheses.

Proposition 3.4. *Let (M, g) be a complete Riemannian manifold, $V : M \to R$ a map of class C^2 and assume that M admits a strongly convex function $F : M \to \mathbb{R}$ (see Definition A.2) such that:*

- *F has a minimum point in M;*
- *F is non-increasing on each flow line of ∇V.*

If p and q are non-conjugate on $[0, T]$ relatively to (2.1), then there are only a finite number of solutions of the dynamical system (2.1) satisfying the boundary conditions (2.3).

Proof. The proof will be obtained as an application of Proposition 3.2, by showing that, under our assumptions, all the solutions of (2.1) and (2.3) must remain inside some compact subset of M.

To this aim, first observe that since F has a minimum point in M, from the strict convexity of F it follows that every sublevel $F^c = F^{-1}(]-\infty, c])$ is compact in M (Proposition A.4). Secondly, observe that the non-increasing assumption for F on the flow lines of ∇V, i.e.,

$$g(\nabla F, \nabla V) \leq 0$$

on M, implies that $F \circ x : [0, T] \to \mathbb{R}$ is strictly convex for every non-constant solution $x : [0, T] \to M$ of (2.1). For, given such a solution x, one computes immediately:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} F(x(t)) = -g(\nabla F(x(t)), \nabla V(x(t))) + \mathrm{Hess}_{x(t)} F(\dot{x}(t), \dot{x}(t)), \tag{3.6}$$

since F is strongly convex and F is non-increasing on each flow line of ∇V, i.e.,

$$-g(\nabla F(x), \nabla V(x)) \geq 0,$$

the right-hand side of (3.6) is strictly positive except possibly at those instants when $\dot{x}(t) = 0$. Now, observe that every non-constant solution x of (2.1) only admits isolated zeros of the derivative $\dot{x}$.

Since convex functions on closed intervals attains their maximum at either one of the endpoints, it follows that every solution of (2.1) and (2.3) must have image in the compact subset F^d, where $d = \max\{F(p), F(q)\}$. This concludes the proof. □

Remark 3.5. From the proof of Proposition 3.4 it is clear that the same conclusion holds if one requires the weaker assumption that F be non-decreasing along the flow lines of ∇V only outside some compact subset of M.

More generally, the finiteness result of Proposition 3.4 holds under the assumption of existence of a (non-necessarily differentiable) proper function $F : M \to R$ such that $F \circ x$ is strictly convex for every (non-constant) solution x of (2.1).

As a corollary of Proposition 2.6 and 3.4 we obtain the following corollary.

Corollary 3.6. *Under the assumption of* Proposition 3.4, *the set* σ_∞ (*recall* (2.6)) *has null measure*.

Proof. By Proposition 3.4, σ_∞ is contained in the set of points of M that are conjugate to p on $[0, T]$ relatively to (2.1); by Proposition 2.6 the latter has null measure. □

4. Some examples

Example 4.1. If the potential $V \equiv 0$, then the dynamical system (2.1) reduces to the geodesic equation on the Riemannian manifold (M, g). In this case, Proposition 3.4 gives us a finiteness result for geodesics joining two fixed non-conjugate points in a manifold admitting a strictly convex function F. This fact was proven in [11] under the more restrictive assumption that F is strongly convex. This result can be used in General Relativity to the study of the so-called *gravitational lensing effect* (see for instance [11,12]); in geometrical terms, this phenomenon corresponds to the existence of multiple lightlike geodesics joining a fixed event p and a fixed observer γ in a Lorentzian manifold. In this context, Proposition 3.2 tells us that if the Lorentzian metric is *static* and if the underlying Riemannian manifold admits a strictly convex function, then the gravitational lensing effect produces only a finite number of images of the light source provided that the source and the observer are not conjugate by lightlike geodesics.

Generalizations of this result to the case of non-static Lorentzian metrics is possible by developing a theory analogue to that of Section 3 for general Lagrangians action functionals of the form $\mathcal{L}(x) = \int_a^b L(t, x, \dot{x})\,\mathrm{d}t$. The case of *stationary* (i.e., time-independent) Lorentzian metrics is studied in [12].

Example 4.2. In the special case that (M, g) is the Euclidean space $\mathbb{R}^n$, then there are several criteria to establish when the dynamical system (2.1) is such that all its solutions satisfying the two-point boundary condition remain inside a compact set. For instance, assume that $\|\nabla V\|$ is bounded:

$$\|\nabla V\| \le M. \tag{4.1}$$

Let $x : [0, T] \to \mathbb{R}^n$ be a solution of (2.1) and (2.3), with $q = (q_1, \dots, q_n)$; denote by $x_i : [0, T] \to \mathbb{R}$ the ith coordinate of x and let $\tau_i \in [0, T]$ be a maximum point for $|x_i|$. We have

$$\begin{aligned}|q_i - x_i(\tau_i)| &= \left|\int_{\tau_i}^{T} \dot{x}_i(s)\right| = \left|\int_{\tau_i}^{T}\left[\int_{\tau_i}^{s} \ddot{x}_i(r)\,\mathrm{d}r\right]\mathrm{d}s\right| \\ &\le \int_{\tau_i}^{T}\int_{\tau_i}^{s} \|\nabla V(x(r))\|\,\mathrm{d}r\,\mathrm{d}s \le MT^2.\end{aligned}$$

Hence, all the solutions of (2.1) and (2.3) have image in the closed ball centered at q and of radius $MT^2\sqrt{n}$. Clearly, the same argument works under milder assumptions on the growth of ∇V in replacement of the boundedness assumption (4.1).

Using the above criterion and Proposition 3.2, we obtain that, given the dynamical system (2.1) in $\mathbb{R}^n$, under suitable assumptions on the growth of $\|\nabla V\|$, then there are only a finite number of solutions joining two non-conjugate points. An example is given by the *pendulum equation* $\ddot{x} = -\sin x$ in $\mathbb{R}$; the corresponding focal decomposition of $\mathbb{R}^2$ is studied in [19]. The remarkable fact about the focal decomposition of $\mathbb{R}^2$ determined by the pendulum equation [20, Fig. 1, p. 519] is that, as it was observed in [20], the very same decomposition of $\mathbb{R}^2$ appears independently in a totally different context on a recent paper in Quantum Statistical Mechanics [3].

Remark 4.3. Given a differential equation

$$\ddot{x} = f(t, x, \dot{x}), \tag{4.2}$$

in $\mathbb{R}^n$, some authors (see for instance [20]) gives the following definition. The equation (4.2) is said to be *regular* if for any compact subset $K_1 \subset \mathbb{R}^n$ there exists a compact subset $K_2 \subset \mathbb{R}^n$ such that all the solutions of (4.2) satisfying $x(a), x(b) \in K_1$ remain inside K_2. For instance, when f is bounded then the same argument of Example 4.2 shows that (4.2) is regular.

It is easy to prove the following criterion to establish the regularity of (2.1).

Lemma 4.4. *Given the dynamical system* (2.1) *in the complete Riemannian manifold* (M, g), *assume that*:

- $\lim_{x\to\infty} V(x) = -\infty$;
- *V does not have critical points outside some compact subset of M*;
- *the Hessian* Hess V *is negative semi-definite on the orthogonal distribution* $\nabla V^{\perp}$ *outside some compact subset of M.*

Then all the solutions $x : [0, T] \to M$ *of* (2.1) *that satisfy the boundary conditions* (2.3) *have image inside a compact subset of M.*

Proof. Since $\lim_{x\to\infty} V(x) = -\infty$, then the closed sets

$$V_c = \{x \in M : V(x) \geq c\}$$

are compact for all $c \in \mathbb{R}$; we will prove that all the solutions of (2.1) and (2.3) remain in some V_c. To this aim, let $x : [0, T] \to M$ be such a solution and let $\tau \in [0, T]$ be a minimum point for $g(t) = V(x(t))$. We can clearly assume that $\tau \in]0, T[$. Let $c \in \mathbb{R}$ be large enough so that $\nabla V \neq 0$ and Hess V is negative semi-definite on the orthogonal distribution $\nabla V^{\perp}$ outside V_c. We have $0 = g'(\tau) = g(\nabla V(x(\tau)), \dot{x}(\tau))$, so that $\dot{x}(\tau) \in \nabla V(x(\tau))^{\perp}$; on the other hand, $x(\tau) \notin V_c$, then

$$g''(\tau) = \mathrm{Hess}_x\, V(\dot{x}(\tau), \dot{x}(\tau)) - g(\nabla V(x(\tau))), \nabla V(x(\tau)) < 0,$$

against the assumption that τ is a minimum for g. This concludes the proof. □

For dynamical systems in Euclidean spaces, we have the following regularity criterion, which is basically well known (see [20, Remark 2, p. 544] for the one-dimensional and time-dependent case).

Lemma 4.5. *Consider the dynamical system* (2.1) *in* $\mathbb{R}^n$, *with* $V : \mathbb{R}^n \to \mathbb{R}$ *a function of class* C^2 *satisfying the condition*:

$$x_i \frac{\partial V}{\partial x_i} < 0 \tag{4.3}$$

for all $i = 1, \ldots, n$ *and for all* x_i *with* $|x_i|$ *sufficiently large. Then all the solutions of* (2.1) *satisfying the boundary conditions* (2.3) *have image in a compact subset of* $\mathbb{R}^n$.

Proof. Let $x = (x_1, \ldots, x_n) : [0, T] \to \mathbb{R}^n$ be a solution of (2.1) satisfying (2.3); for all $i = 1, \ldots, n$, let $\tau_i \in [0, T]$ be a maximum point for $x_i(t)$. We can clearly assume $\tau_i \in]0, T[$

and $x_i(\tau_i) \neq 0$. If $x(\tau_i) > 0$, the value of $x_i(\tau_i)$ cannot be *too large*, because otherwise, by (4.3), it would be $\ddot{x}_i(\tau_i) = -\partial V/\partial x_i(x(\tau_i)) > 0$ and $x(\tau_i)$ could not be a maximum for x_i. Similarly, if $x(\tau_i) < 0$, then $-x(\tau_i)$ cannot be very large, because otherwise by (4.3) $\ddot{x}_i(\tau_i) = -\partial V/\partial x_i(x(\tau_i)) < 0$ and $x(\tau_i)$ could not be a minimum for x_i. This concludes the proof. □

Examples of potentials V satisfying the assumptions of Lemma 4.5 are all polynomial functions on $\mathbb{R}^n$ containing only even powers of the variables and whose coefficients are negative.

Example 4.6. If (M, g) admits a strongly convex function F, then Proposition 3.4 can be applied to the case that $V = -F$. Actually, in this case the convexity assumption can be weakened, and we obtain easily the following finiteness result for concave potentials. We say that a map $G : M \to \mathbb{R}$ is (*strongly*) *concave* if $-G$ is (strongly) convex.

Proposition 4.7. *Let* (M, g) *be a complete Riemannian manifold and* $V : M \to \mathbb{R}$ *be a map of class* C^2 *which has an isolated maximum and which is strongly concave. Then, the dynamical system* (2.1) *admits only a finite number of solutions* $x : [0, T] \to M$ *satisfying* (2.3) *provided that p and q are non-conjugate on* $[0, T]$ *relatively to* (2.1).

Proof. The proof of Proposition 3.4 can be repeated *verbatim* by setting $F = -V$, except for the following detail. Given a solution $x : [0, T] \to M$ of (2.1), the strict convexity of $F \circ x$ is now proven, rather than by the strong convexity of F, by observing that $\mathrm{Hess}_x F(\dot{x}, \dot{x}) \geq 0$, while $-g(\nabla F(x), \nabla V(x)) = g(\nabla V(x), \nabla V(x)) > 0$ except possibly when x passes through the unique critical point of V. □

5. Generically, the number of solutions of (2.1) and (2.3) is odd

One of the most powerful tools for *counting* the number of critical points of a C^2 map is given by Morse theory. In this section we use a classical result of infinite dimensional Morse theory to conclude that the set of solutions of (2.1) and (2.3), under the non-conjugacy assumption for p and q is either infinite or it has an odd number of elements.

The main technical assumption in order to develop the Morse theory on an infinite dimensional Hilbert manifold is the Palais–Smale condition; for this reason we will make the assumption of subquadratic growth of the potential function V. The following *odd number theorem* is proven under the assumption that the underlying Riemannian manifold M has *trivial topology*, i.e., that it is contractible. This is the case, for instance, when M admits a non-trivial convex function with a minimum point (see Proposition A.3).

Recall that, given a topological space X and a ring R, the ith *Betti number* $\beta_i(X; R)$ of X is the R-dimension of the ith singular homology module of X with coefficients in R; the *Poincaré formal power series* $\mathcal{P}_\lambda(X; R)$ of X in R in the variable $\lambda \in R$ is the series:

$$\mathcal{P}_\lambda(X; R) = \sum_{i=1}^{\infty} \beta_i(X; R)\, \lambda^i.$$

Proposition 5.1. *Suppose that* (M, g) *is a complete and contractible Riemannian manifold,* $p \in M$*, and let* $V : M \to \mathbb{R}$ *be a* C^2 *map which has subquadratic growth, i.e., there exists constants* $C_1, C_2 \in \mathbb{R}^+$ *and* $\alpha \in]0, 2[$*, such that*

$$V(x) \leq C_1 + C_2 \cdot \mathrm{dist}(x, x_0)^\alpha$$

for all $x \in M$ *and for some fixed* x_0 *in* M.

Then, for all even i*, the set* σ_i (*recall* (2.6)) *has null measure in* M.

Proof. By Proposition 2.6, it suffices to show that, under our assumptions, there are either infinitely many or an odd number of solutions of (2.1) satisfying (2.3) if p and q are not conjugate on $[0, T]$ relatively to (2.1).

To this aim, assume that p and q are not conjugate, so that the Lagrangian action functional $\mathcal{L}_T$ (defined in (2.4)) is a Morse function on the complete Hilbert manifold $\Omega_{p,q}([0, T], M)$. Since V has subquadratic growth, then $\mathcal{L}_T$ satisfies the Palais–Smale condition on $\Omega_{p,q}([0, T], M)$ and it is bounded from below. Denote by $\mathcal{S}_{p,q}$ the set of solutions of (2.1) satisfying (2.3); for $x \in \mathcal{S}_{p,q}$ denote by $\mu(x)$ the number of conjugate instants along x (recall Definition 2.7). Then, we can write the classical Morse relations for the critical points of $\mathcal{L}_T$ (see for instance [16,17]), that are expressed by the following equality of formal power series in $\lambda \in \mathbb{R}$:

$$\sum_{x \in \mathcal{S}_{p,q}} \lambda^{\mu(x)} = \mathcal{P}_\lambda(\Omega_{p,q}([0, T], M); \mathbb{R}) + (1 + \lambda) Q(\lambda), \tag{5.1}$$

where $P_\lambda(\Omega_{p,q}([0, T], M); \mathbb{R})$ is the Poincaré formal power series of $\Omega_{p,q}([0, T], M)$ in $\mathbb{R}$, and $Q(\lambda)$ is a formal power series in λ with coefficients in $\mathbb{N} \bigcup \{+\infty\}$.

Since M is contractible, then so is $\Omega_{p,q}([0, T], M)$, hence $\beta_i(\Omega_{p,q}([0, T], M); \mathbb{R}) = 0$ for all $i \geq 1$ and $\beta_0(\Omega_{p,q}([0, T], M); \mathbb{R}) = 1$. Then, setting $\lambda = 1$ in (5.1), we obtain the following equality:

$$\text{cardinality of } \mathcal{S}_{p,q} = 1 + 2\, Q(1).$$

Since $Q(1) \in \mathbb{N} \bigcup \{+\infty\}$, the above equality says that the cardinality of $\mathcal{S}_{p,q}$ is either infinite or odd. This concludes the proof. □

If the manifold M is not contractible, then, under the subquadratic growth assumption for the potential V, the number of solutions of the dynamical system (2.1) satisfying (2.3) is infinite; this is easily proven using the Ljusternik–Schnirelman theory. Recall that the *Ljusternik–Schnirelman category* $\mathrm{cat}(X)$ of a topological space X is the minimal cardinality (possibly infinite) of a covering of X by closed and contractible subsets. The main result of the Ljusternik–Schnirelman theory is that any C^1 functional $f : X \to \mathbb{R}$ on a complete Banach manifold X that satisfies the Palais–Smale condition at every level and that is bounded from below, has at least $\mathrm{cat}(X)$ critical points in X.

Proposition 5.2. *Assume that* (M, g) *is a complete Riemannian manifold with* M *non-contractible, and let* $V : M \to \mathbb{R}$ *be a map of class* C^2 *having subquadratic growth. Then,* *for all $p, q \in M$, there are infinitely many distinct solutions of* (2.1) *satisfying the boundary conditions* (2.3).

Proof. By a well known result of Fadell and Husseini [7], if M is not contractible the Ljusternik–Schnirelman category of $\Omega_{p,q}([0, T], M)$ is infinite. The conclusion follows immediately from the Ljusternik–Schnirelmann theory applied to the functional $\mathcal{L}_T$. □

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Appendix A. Convex functions on Riemannian manifolds

The theory of convex functions has its natural setting in Riemannian geometry. In this Appendix we recall a few basic facts and examples of Riemannian manifolds admitting convex functions; a very comprehensive reference for the interested reader is the book by Udrişte [24]. Throughout the section (M, g) will denote a complete Riemannian manifold.

Definition A.1. A subset $A \subset M$ is said to be *totally convex* if it contains every geodesic segment between any two of its points. A function $F : A \subset M \to \mathbb{R}$ defined on a totally convex subset A of M is said to be convex if $F \circ \gamma : [a, b] \to \mathbb{R}$ is *convex* for all geodesic $\gamma : [a, b] \to A$; F is *strictly convex* if $F \circ \gamma$ is strictly convex for every non-constant geodesic γ in A.

Whenever the domain A of a convex function is not specified it will be implicitly assumed that $A = M$.

For a function $F : M \to \mathbb{R}$ of class C^2, the convexity is equivalent to the positive semi-definiteness of the Hessian Hess F; clearly, if Hess F is positive definite in the complement of a finite set, then F is strictly convex. We give the following definition.

Definition A.2. Given a function of class $C^2 F : M \to \mathbb{R}$, we say that F is *strongly convex* if there exists a (continuous) function $\lambda : M \to \mathbb{R}^+$ such that

$$\mathrm{Hess}_x\, F(x)(v, v) \geq \lambda(x) g(v, v), \quad \forall x \in M, \quad \forall v \in T_x M.$$

It is easy to see that if $F : M \to \mathbb{R}$ is a convex function which is bounded from above, then F is constant; it follows in particular that compact Riemannian manifolds do not admit non-constant convex functions. More generally, if a complete Riemannian manifold (M, g) admits a non-constant convex function, then its volume is infinite [23]. As to the regularity of a convex function F, we have that $F : A \to \mathbb{R}$ is continuous at the interior points of A; moreover, a convex function on M is Lipschitz continuous on every compact subset of M.

For functions F of class C^1 or C^2 on M the notion of convexity can be given respectively in terms of the gradient ∇F and the Hessian Hess F of F. A C^1 map $F : A \to \mathbb{R}$ is convex if and only if for all $x, y \in A$ and all geodesic segment $\gamma : [a, b] \to A$ from x to y it is $F(y) \geq F(x) + g(\nabla F(x), \dot{\gamma}(a))$; F is strictly convex if the inequality is strict when $x \neq y$. A C^2 map $F : A \to \mathbb{R}$ is convex if and only if its Hessian Hess F is positive semi-definite in A.

The existence of a non-constant convex function does not have implications on the topological structure of the manifold, except for the property of non-compactness. Namely, by a result of Green and Shioama [9], for every non-compact manifold M there exists a complete Riemannian metric g on M and a non-constant smooth function $F : M \to \mathbb{R}$ which is convex in (M, g). On the other hand, the existence of a non-constant function satisfying further properties, like for instance admitting an isolated minimum point, or such that its sublevels are compact, implies substantial information about the topological structure and the metrical structure of the manifold (see for instance [1]). For instance, it is not hard to prove the following proposition.

Proposition A.3. *If (M, g) admits a strictly convex function that has a strict local minimum, then M is diffeomorphic to $\mathbb{R}^n$.*

Given a simply connected Riemannian manifold (M, g) with non-positive sectional curvature, then for every x_0 the map $x \mapsto \mathrm{dist}(x, x_0)^2$ is convex. The *height function* $(x, y, z) \mapsto z$ defined on the paraboloid $z = x^2 + y^2$ is an example of a strictly convex smooth function on a manifold with positive scalar curvature. A class of examples of Riemannian manifolds admitting smooth convex functions with non-isolated zeroes are the tangent bundles TM of Riemannian manifolds (M, g), when endowed with the Sasaki metric g_{s}: the map $v \to g(v, v)$ is convex in (TM, g_{s}).

Concerning the *growth at infinity* of strictly convex functions, we have the following proposition.

Proposition A.4. *Let $F : M \to \mathbb{R}$ be a strictly convex function of class C^1. Assume that F has an isolated critical point. Then, this critical point is unique, it is a minimum point for F, and* $\lim_{x\to\infty} F(x) = +\infty$, *i.e., each sublevel* $F^c = F^{-1}(]-\infty, c])$ is compact in M.

Proof. See for instance [11]. □

We will briefly describe some well known constructions of convex functions.

Recall that a *pole* of (M, g) is a point $x_0 \in M$ such that $\exp_{x_0} : T_{x_0}M \to M$ is a global diffeomorphism. For instance, if M is simply connected and it has non-positive sectional curvature, then by the Hadamard theorem every point of M is a pole.

Example A.5. Given a manifold with a pole x_0, then for every $x \in M$ there exists a unique geodesic $\gamma_x : [0, 1] \to M$ from x_0 to x; the *radial curvature relative to the pole* x_0 of M at

x is defined to be the maximum of the sectional curvatures of all planes in T_xM containing the direction $\dot\gamma_x(1)$.

If (M, g) admits a pole x_0 such that its relative radial curvature is everywhere non-positive, then the function $F(x) = \mathrm{dist}(x, x_0)^2$ is strictly convex. Observe that, in this case, x_0 is an isolated minimum point for F.

Convex functions exist also on manifolds having non-negative sectional curvature. Recall that a *ray* in (M, g) is a geodesic (parameterized by arc length) $\gamma : [0, +\infty[\to M$ such that $\gamma|_{[0,T]}$ is minimal for all $T > 0$, i.e., such that $\mathrm{dist}(\gamma(0), \gamma(T)) = T$ for all $T > 0$. Through every point of a non-compact complete Riemannian manifold there exists a ray.

Example A.6. Given a ray $\gamma : [0, +\infty[\to M$ in M, the *Busemann function* [4,6] η_γ associated to γ is given by

$$\eta_\gamma(x) = \lim_{t\to+\infty}[t - \mathrm{dist}(x, \gamma(t))].$$

If (M, g) is a complete, non-compact Riemannian manifold whose sectional curvature is non-negative outside a compact subset of M, then each η_γ is convex [10]. In particular, in this situation, the supremum η_p of all Busemann functions associated to the rays starting at a fixed point $p \in M$ is a convex function that is non-constant on every non-constant geodesic of M; moreover, the sublevels of η_p are compact. Observe that given a convex function $F : M \to \mathbb{R}$ which is non-constant on every non-constant geodesic, then for all strictly increasing and strictly convex function $\Psi : \mathbb{R} \to \mathbb{R}$ the composition $\Psi \circ F$ is strictly convex on M. Observe also that, in general, η_p is not differentiable; for instance, if (M, g) is a Euclidean space and $p \in M$ is the origin, then $\eta_p(x) = \|x\|$.

It is easy to see that if $F : A \to \mathbb{R}$ is a convex function which is not constant on every non-constant geodesic in A and if $\Psi : \mathbb{R} \to \mathbb{R}$ is a strictly convex function, then the composition $\Psi \circ F$ is strictly convex in A.

Killing fields give rise to convex functions provided that the associated Killing curvature of M is non-positive; recall that if Y is a Killing vector field on (M, g), then the associated *Killing curvature* at $p \in M$ is the maximum of the sectional curvatures of all planes in T_pM containing the direction $Y(p)$.

Example A.7. If Y is a Killing vector field on (M, g) whose Killing curvature is non-positive, then the map $F(x) = (1/2)g(Y(x), Y(x))$ is convex. Clearly, the minimum points of F are the zeroes of Y; more generally, the critical points of F are precisely the points $x \in M$ such that the integral curve of Y through x is a geodesic.

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